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ON SYMMETRIC BI-MULTIPLIERS OF SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduce the notion of symmetric bi-multiplier of subtraction algebra and investigate some related properties. Also, we prove that if D is a symmetric bi-multiplier of X, then D is an isotone symmetric bi-multiplier of X.

1. Introduction

B. M. Schein ([4]) considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of ([1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka ([6]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we introduce the notion of symmetric bi-multiplier of subtraction algebra and investigated some related properties. Also, we prove that if D is a symmetric bi-multiplier of X, then D is an isotone symmetric bi-multiplier of X.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

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(S1) $x-(y-x)=x$;
(S2) $x-(x-y)=y-(y-x)$;
(S3) $(x-y)-z=(x-z)-y$.

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a-(a-b)$; the complement of an element $b \in [0,a]$ is $a-b$; and if $b,c \in [0,a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a-((a-b)\wedge(a-c)) \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

In a subtraction algebra X, the following are true (see [4]):

(p1) $(x-y)-y=x-y$ for any $x,y \in X$.
(p2) $x-0=x$ and $0-x=0$ for any $x \in X$.
(p3) $(x-y)-x=0$ for any $x,y \in X$.
(p4) $x-(x-y) \leq y$ for any $x,y \in X$.
(p5) $(x-y)-(y-x)=x-y$ for any $x,y \in X$.
(p6) $x-(x-(x-y))=x-y$ for any $x,y \in X$.
(p7) $(x-y)-(z-y) \leq x-z$ for any $x,y,z \in X$.
(p8) $x \leq y$ for any $x,y \in X$ if and only if $x=y-w$ for some $w \in X$.
(p9) $x \leq y$ implies $x-z \leq y-z$ and $z-y \leq z-x$ for all $z \in X$.
(p10) $x,y \leq z$ implies $x-y=x \wedge (z-y)$ for any $x,y,z \in X$.
(p11) $(x \wedge y)-(x \wedge z) \leq x \wedge (y-z)$ for any $x,y,z \in X$.
(p12) $(x-y)-z=(x-z)-(y-z)$. for any $x,y,z \in X$.

A mapping d from a subtraction algebra X to a subtraction algebra Y is called a *morphism* if $d(x-y)=d(x)-d(y)$ for all $x,y \in X$. A self map d of a subtraction algebra X which is a morphism is called an *endomorphism*.

LEMMA 2.1. *Let X be a subtraction algebra. Then the following properties hold:*
(1) *$x \wedge y = y \wedge x$ for every $x,y \in X$.*
(2) *$x-y \leq x$ for all $x,y \in X$.*

LEMMA 2.2. *Every subtraction algebra X satisfies the following property.*

$$(x-y)-(x-z) \leq z-y$$

for all $x, y, z \in X$.

Definition 2.3. Let X be a subtraction algebra and Y a non-empty subset of X. Then Y is called a *subalgebra* if $x - y \in Y$ whenever $x, y \in Y$.

Definition 2.4. Let X be a subtraction algebra. A mapping $D : X \times X \to X$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in X$.

Definition 2.5. Let X be a subtraction algebra. A mapping $d(x) = D(x, x)$ is called *trace* of $D(., .)$ where $D : X \times X \to X$ is a symmetric mapping.

3. Symmetric bi-multipliers of subtraction algebras

In what follows, let X denote a subtraction algebras unless otherwise specified.

Definition 3.1. Let X be a subtraction algebra and D be a symmetric map. A function $D : X \times X \to X$ is called a *symmetric bi-multiplier* on X if it satisfies the following condition

$$D(x \wedge z, y) = D(x, y) \wedge z$$

for all $x, y, z \in X$.

Example 3.2. Let $X = \{0, a, b, c\}$ be a set in which "$-$" is defined by

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

It is easy to check that $(X; -)$ is a subtraction algebra. Define a map $D : X \times X \to X$ by

$$D(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{if } (x,y) = (a,a) \\ b & \text{if } (x,y) = (b,b) \\ c & \text{if } (x,y) = (c,c) \\ 0 & \text{if } (x,y) = (b,a),(a,b) \\ a & \text{if } (x,y) = (a,c),(c,a) \\ b & \text{if } (x,y) = (b,c),(c,b)\ . \end{cases}$$

Then it is easily checked that D is a symmetric bi-multiplier of X.

PROPOSITION 3.3. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. Then $D(0,x) = 0$ for all $x \in X$.*

Proof. For all $x \in X$, we get

$$\begin{aligned} D(0,x) &= D(0 \wedge 0, x) \\ &= D(0,x) \wedge 0 = 0. \end{aligned}$$

This completes the proof. □

PROPOSITION 3.4. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. Then $d(x) \leq x$ for all $x \in X$.*

Proof. Since $x \wedge x = x$, we have

$$\begin{aligned} d(x) &= D(x,x) \\ &= D(x \wedge x, x) = D(x,x) \wedge x \\ &= d(x) \wedge x \end{aligned}$$

for all $x \in X$. Therefore $d(x) \leq x$ for all $x \in X$ by (S2) and (p4). □

PROPOSITION 3.5. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. Then $D(x,y) \leq x$ and $D(x,y) \leq y$ for all $x, y \in X$.*

Proof. Since $x \wedge x = x$, we have

$$\begin{aligned} D(x,y) &= D(x \wedge x, y) \\ &= D(x,y) \wedge x \end{aligned}$$

for all $x \in X$. Therefore $D(x,y) \leq x$ for all $x, y \in X$ by (S2) and (p4). Similarly, we see that $D(x,y) \leq y$ for all $x, y \in X$. □

THEOREM 3.6. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. Then d is an isotone mapping on X.*

Proof. Let $x \leq y$. Then $x - y = 0$. Hence we have

$$\begin{aligned} d(x) &= D(x,x) = D(x \wedge y, x \wedge y) \\ &= D(y \wedge x, x \wedge y) = D(y, x \wedge y) \wedge x \\ &= D(y \wedge x, y) \wedge x = (D(y,y) \wedge x) \wedge x \\ &\leq D(y,y) \wedge x \leq D(y,y) = d(y). \end{aligned}$$

This implies that d is an isotone mapping on X. □

Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. For a fixed element $a \in X$, define a map $d_a : X \to X$ by $d_a(x) = D(x,a)$ for all $x \in X$.

PROPOSITION 3.7. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. Then the following conditions hold:*
(1) *$d_a(x) = d_a(x) \wedge x$ for every $x \in X$.*
(2) *If $x \leq y$, then $d_a(x) = d_a(x) \wedge y$ for $x, y \in X$.*

Proof. (1) For every $x \in X$, we have

$$\begin{aligned} d_a(x) &= D(x,a) = D(x \wedge x, a) \\ &= D(x,a) \wedge x = d_a(x) \wedge x. \end{aligned}$$

(2) Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$. Hence

$$\begin{aligned} d_a(x) &= D(x,a) = D(x - (x - y), a) \\ &= D(x \wedge y, a) = D(x,a) \wedge y = d_a(x) \wedge y. \end{aligned}$$

This completes the proof. □

PROPOSITION 3.8. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. Then d_a is an isotone mapping on X.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$. Hence

$$\begin{aligned} d_a(x) &= D(x,a) = D(x - (x - y), a) \\ &= D(x \wedge y, a) = D(y \wedge x, a) \\ &= D(y,a) \wedge x \leq D(y,a) = d_a(y). \end{aligned}$$

This implies that d_a is an isotone mapping on X. □

PROPOSITION 3.9. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. Then d_a is regular, that is, $d_a(0) = 0$.*

Proof. Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X.

$$\begin{aligned} d_a(0) &= D(0,a) = D(0 \wedge 0, a) \\ &= D(0,a) \wedge 0 = 0. \end{aligned}$$

This implies that d_a is regular. □

PROPOSITION 3.10. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. Then $d_a(x \wedge y) \leq d_a(x)$ for all $x, y \in X$.*

Proof. Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X.

$$\begin{aligned} d_a(x \wedge y) &= D(x \wedge y, a) = D(x,a) \wedge y \\ &= d_a(x) \wedge y \leq d_a(x). \end{aligned}$$

This completes the proof. □

DEFINITION 3.11. Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X. If $x \leq w$ implies $D(x,y) \leq D(w,y)$ for every $y \in X$, D is called an *isotone symmetric bi-multiplier* of X.

THEOREM 3.12. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier on X. Then D is an isotone symmetric bi-multiplier of X.*

Proof. Let $x \leq y$. Then $x - y = 0$. Hence we have

$$\begin{aligned} D(x,z) &= D(x - (x - y), z) = D(x \wedge y, z) \\ &= D(y \wedge x, z) = D(y,z) \wedge x \\ &\leq D(y,z) \end{aligned}$$

for all $z \in X$. This implies that D is an isotone symmetric bi-multiplier of X. □

Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. Define a set $Fix_d(X)$ by

$$Fix_d(X) = \{x \in X \mid d(x) = x\}.$$

PROPOSITION 3.13. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. If $x \in Fix_d(X)$ and $y \in X$, then $x \wedge y \in Fix_d(X)$.*

Proof. Let $x \in Fix_d(X)$. Then $d(x) = x$. Hence

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) = D(x, x \wedge y) \wedge y \\ &= D(x \wedge y, x) \wedge y = (D(x, x) \wedge y) \wedge y \\ &= (d(x) \wedge y) \wedge y = (x \wedge y) \wedge y \\ &= x \wedge y, \end{aligned}$$

since $x \wedge y \leq y$ for all $x, y \in X$. This implies that $x \wedge y \in Fix_d(X)$. □

Proposition 3.14. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. Then $Fix_d(X)$ is a down closed set, that is, $y \in Fix_d(X)$ and $x \leq y$ implies $x \in Fix_d(X)$.*

Proof. Let $y \in Fix_d(X)$ and $x \leq y$. Then $d(y) = y$. Hence

$$\begin{aligned} d(x) &= D(x, x) = D(x \wedge y, x \wedge y)) = D(y \wedge x, y \wedge x) \\ &= D(y, y \wedge x) \wedge x = (D(y \wedge x, y) \wedge x \\ &= (D(y, y) \wedge x) \wedge x = (d(y) \wedge x) \wedge x = (y \wedge x) \wedge x \\ &= x. \end{aligned}$$

This implies that $x \in Fix_d(X)$. □

Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. Define a set $Kerd$ by

$$Kerd = \{x \in X \mid d(x) = 0\}.$$

Proposition 3.15. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. If $y \in Kerd$ and $x \in X$, then $x \wedge y \in Kerd$.*

Proof. Let $y \in Kerd$. Then $d(y) = 0$.

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) = D(x, x \wedge y) \wedge y \\ &= D(x \wedge y, y) \wedge y = D(y \wedge x, y) \wedge y \\ &= (D(y, y) \wedge x) \wedge y = (0 \wedge x) \wedge y \\ &= 0 \end{aligned}$$

for all $x \in X$. This implies $x \wedge y \in Kerd$. □

Proposition 3.16. *Let X be a subtraction algebra and let D be a symmetric bi-multiplier of X with the trace d. Then $Kerd$ is a down closed set, that is, $x \in Kerd$ and $y \leq x$ implies $y \in Kerd$.*

Proof. Let $x \in Kerd$ and $y \leq x$. Then $d(x) = 0$ and $y - x = 0$. Hence

$$\begin{aligned} d(y) &= D(y,y) = D(x \wedge y, x \wedge y) \\ &= D(x, x \wedge y) \wedge y = (D(x,x) \wedge y) \wedge y \\ &= (d(x) \wedge y) \wedge y = (0 \wedge y) \wedge y = 0. \end{aligned}$$

This implies that $y \in Kerd$. □

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